

Actual computational time-cost of the Quantum Fourier Transform in a quantum computer using nuclear spins

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There have been many proposed methods for the practical implementation of quantum computing. Now quantum computation has reached the turning point from being a conceptual system to becoming a physical one. In this paper, we discuss a practical elementary gate and the actual computational time-cost of the QFT in two physical implementations, namely the bulk spin resonance computer and the Spin Resonance Transistor. We show that almost all universal gates require different times for operation. The actual time-cost of the QFT is $O(n2^n)$ for large n . This differs drastically from the reported cost $O(n^2)$ based on ideal quantum computation.

Recent technological development has stimulated proposals for many quantum computers: Bulk Spin Resonance(BSR) [1–3], trapped ions [4], cavityQED [5], Josephson junctions [6], coupled quantum dots [7] and the Spin Resonance Transistor(SRT) [8,9]. BSR has been implemented experimentally in some organic molecules by the use of conventional nuclear magnetic resonance(NMR) equipment [1–3]. the SRT is attractive from the viewpoint of it's integration ability and it's compatibility with silicon technology.

Complexity analysis classifies quantum algorithms according to a function that describes how a computational cost incurred in solving a problem scales up as larger problems are considered [10]. The computational cost of a quantum algorithm has usually been estimated as the sum of the universal gates required in such ideal mathematical models as the Quantum Turing Machine(QTM) and the quantum circuit. The computational complexity is effective in estimating the essential performance of an algorithm to factor out the variations in performance experienced by different makes of computers with different amounts of Random Access Memory(RAM), swap space, and processor speeds. The above cost is proportional to an actual time-cost in the physical implementation where all quantum operations can be achieved in the same time. However, if the implementation being considered takes a different time for each quantum gate, there is a possibility that the actual time-cost will have a different behavior from the ideal cost. A hardware dependent time-cost is important to experimentalists who research into practical implementations. In this paper, we focus on the actual computational time-cost of the Quantum Fourier Transform (QFT) in two physical implementations that utilise nuclear spins: BSR and the SRT.

For the following discussion, we assumed that the

quantum computers being considered are constructed from an array of n quantum registers labeled by j ($1 \leq j \leq n$) in one dimension. They calculate n qubit data which corresponds to $N = 2^n$ states. We defined the following matrices. The suffixes on each matrix represent the quantum registers on which it operates. In the definitions of $C_{j,k}(\theta)$ and $D_{j,k}(\theta)$, the first and the second suffixes represent the target bit and the controlled bit respectively.

$$H_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_{yj}(\theta) = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix}$$

$$R_{zj}(\alpha) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}, \quad C_{j,k}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix}$$

$$\Phi_j(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}, \quad D_{j,k}(\theta) = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{-i\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix}$$

Firstly, we discuss the practical elementary gate in these implementations. The practical elementary gate is defined as a quantum gate that can be achieved directly by physical phenomena in the implementation being considered.

BSR consists of quantum registers that are the nuclear spins of each atom in an organic molecule [1–3]. External RF-pulses and magnetic fields control the quantum states and the quantum correlations in the following way.

The nuclear spin under a strong magnetic field $\mathbf{B} = (0, 0, B_0)$ is described by the Hamiltonian $H = -\gamma\hbar B_0 I_{jz}$, where γ is the gyro-magnetic ratio for the spin, and I_{jz} is the z component of the j th nuclear spin. The time evolution of the system is $\exp(iHt/\hbar) = \exp(-i\gamma\hbar B_0 t S_{jz})$ and this then corresponds to the rotation on the z -axis $R_{zj}(\theta = \gamma\hbar B_0 t)$. The RF-pulse enables the rotation on the other axis [11]. The time evolution shows that the rotation angle θ can be controlled by two factors: the intensity B_0 , and the duration t , of the external magnetic field. Consequently, there are two control modes. In the intensity control mode, each single qubit rotation takes the same time. In the duration control mode however, each single qubit phase rotation takes a different time, which is proportional to the rotation angle.

The exchange interaction between the j -th and the k -th registers is described by the Hamiltonian $H_{exch.} =$

$J_{jk}I_{jz}I_{kz}$, where J_{jk} is the exchange coupling constant. The time evolution of the system is described by

$$\exp(iJ_{jk}tI_{jz}I_{kz}/\hbar) = D_{j,k}(\theta = J_{jk}t/2\hbar). \quad (1)$$

Sequence(1) shows that the exchange interaction can control the phase rotation angle. Any external fields, however, cannot control the interaction directly, because atoms in a molecule always interact with each other. The refocusing technique can control the angle θ effectively [11]. Only the time duration between the refocusing pulses determines the angle θ . The operation $D_{j,k}(\theta)$, therefore, takes a time that is proportional to the rotation angle θ . An actual operation such as $D_{j,k}$ is effective only for adjacent registers because the exchange interaction $J_{j,k}$ between non-adjacent registers is very small.

The SRT is composed of quantum registers which are the nuclear spins of arrayed phosphorus ions in silicon, with a globally static magnetic field B and an AC magnetic field B_{AC} [8]. The implementation consists of two gates on the surface: the A-gate above each ion and the J-gate between adjacent ions.

The A-gate controls the strength of hyperfine interactions and the resonance frequency of the nuclear spin beneath it. A globally applied magnetic field B_{AC} flips nuclear spins resonant with the field by the same process that occurs in BSR. In this case, only the duration of the resonance determines the rotation angle. It implies that each single qubit phase rotation always takes a different time.

The electron wave function extends over a large distance and makes an effective electron-mediated coupling for two nuclear spins sharing it in semiconductors. The J-gate controls the overlap of the electron wave functions bounded to two adjacent phosphorus atoms, and hence the electron-mediated exchange coupling $J_{j,k} = J(t)$ in the time evolution(1) directly. It therefore controls the phase rotation angle in the operation $D_{j,k}$. The operation $D_{j,k}$ in this implementation also operates only for adjacent registers.

In both implementations, all single qubit phase rotations and controlled phase rotations are practical elementary gates. They can make the quantum XOR in the sequence below, ordered from right to left [1]

$$\begin{aligned} & \sqrt{-i}XOR(j, k) \\ &= R_{yj}(-\frac{\pi}{2})R_{zk}(-\frac{\pi}{2})R_{zj}(-\frac{\pi}{2})D_{j,k}(\frac{\pi}{4})R_{yj}(\frac{\pi}{2}). \end{aligned} \quad (2)$$

The suffixes j and k represent the target bit and the controlled bit respectively. The sequence(2) shows that the quantum XOR depends on single qubit gates in this technique, and is then concerned with the time required for phase rotation.

Barenco and co-workers showed that other universal gates can be constructed by all single qubit gates and the quantum XOR [12]. Each $n(\geq 1)$ qubit gate required a different number of them [12], and then a different time

for execution. As discussed above, almost all universal gates take a different time in these implementations. It suggests the possibility that the actual time-cost of the quantum algorithm is different from the ideal cost, though the complexity is not affected in the QFT case.

Next we estimate the actual time-cost of the QFT that could be achieved by the above practical elementary gates, by considering the time resolutions of the controlling external fields in these implementations.

The QFT is the transform with base $N = 2^n$, corresponding to the n qubit defined by

$$|x\rangle \rightarrow \frac{1}{\sqrt{q}} \sum_{c=0}^{N-1} e^{\frac{2\pi i c x}{N}} |c\rangle. \quad (3)$$

Shor proposed the algorithm factoring a composite integer by the QFT [13]. The QFT can be constructed by the sequence [13] in the order (from right to left)

$$\begin{aligned} & H_0 C_{0,1}(\theta_1) C_{0,2}(\theta_2) \cdots C_{0,n-1}(\theta_{n-1}) H_1 \cdots H_{n-3} \\ & C_{n-3,n-2}(\theta_1) C_{n-3,n-1}(\theta_2) H_{n-2} C_{n-2,n-1}(\theta_1) H_{n-1} \end{aligned} \quad (4)$$

followed by a bit reversal transformation, where $\theta_j \equiv \pi/2^j$. The n qubit QFT requires $n(n-1)/2$ controlled phase-shifter ($C_{j,k}(\theta_{k-j})$ s and n Hadamard transformation H_j s, and then $n(n+1)/2 \simeq O(n^2)$ [13]. This estimation is based on the complexity analysis method. It coincides with an actual time-cost in the case where all gates required in the sequence(4) are practical elementary ones.

The controlled phase rotation $C_{j,k}(\theta_{k-j})$ can be achieved by the sequence

$$\begin{aligned} & C_{j,k}(\theta_{k-j}) = R_{zk}(-\theta_{k-j+1}) \Phi_k(\theta_{k-j+2}) R_{zj}(-\theta_{k-j+1}) \\ & XOR(j, k) R_{zj}(\theta_{k-j+1}) XOR(j, k). \end{aligned} \quad (5)$$

The intensity control mode can make all operations ($C_{j,k}(\theta_{k-j}), H_j$) take almost the same time. Consequently, the actual time-cost coincides with the ideal cost in this mode.

The duration control mode, however, makes each operation $C_{j,k}(\theta_{k-j})$ require the time τ_{k-j} in proportion to the phase rotation angle θ_{k-j} . The operation $C_{0,n-1}(\theta_{n-1})$ rotates the minimum phase angle θ_{n-1} and takes the minimum time τ_{n-1} of all rotations in the QFT(4). The range of required phase rotations in the QFT(4) increases with 2^n . For example, the time ratio τ_0/τ_{n-1} approximates to $2^{100} \simeq 10^{30}$ in the 100 qubit QFT.

In general, the time resolution t_R controlling the external field is determined by the response time of the system, the delay of the electronic signal and so on. We can only execute in the physical implementations that satisfy the relationship:

$$\tau_0 > \tau_1 > \cdots > \tau_{n-1} \geq t_R. \quad (6)$$

It is important for the actual time-cost estimation to determine how we set up the unit time t_{unit} . The unit

time t_{unit} should be also greater than the time resolution t_R . The QFT's in these implementations have various actual time-costs from $t_{unit} = \tau_0$ to $t_{unit} = \tau_{n-1}$.

If we adopt the maximum rotation time τ_0 as the unit time t_{unit} , the actual time-cost is $O(n)$, since

$$\begin{aligned} \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} \frac{\tau_{k-j}}{\tau_0} &= \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} \frac{\theta_{k-j}}{\theta_0} = \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} 2^{j-k} \\ &= n + 2^{1-n} - 2 \simeq O(n). \end{aligned} \quad (7)$$

On the other hand, the condition $t_{unit} = \tau_{n-1}$ makes the actual time-cost $O(n2^n)$, since

$$\begin{aligned} \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} \frac{\tau_{k-j}}{\tau_{n-1}} &= \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} \frac{\theta_{k-j}}{\theta_{n-1}} \\ &= \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} 2^{n-1+j-k} \\ &= (n-2)2^{n-1} + 1 \simeq O(n2^n). \end{aligned} \quad (8)$$

In this way, the actual time-cost varies from $O(n)$ to $O(n2^n)$, and it depends on which of these is adopted as the unit time t_{unit} . The former time-cost however, is not valid for any n in the following way.

The former condition, $t_{unit} = \tau_0$, means that all phases are always rotated by the external field with constant intensity B for various data of magnitude n . In this case, the minimum time τ_{n-1} decreases exponentially with increasing n . We cannot rotate the phase θ_j to satisfy the condition $\tau_j < t_R$. There exists the upper bound n_b , satisfying the relationship(6) for the intensity B under consideration. The estimated time-cost(7) is valid for any n satisfying $n \leq n_b$. It is, however, not valid for any n which is greater than n_b .

The latter condition $t_{unit} = \tau_{n-1}$ means that the intensity B decreases exponentially with n satisfying the relation(6). In this case, we can rotate all phase angles θ_j ($0 \leq j \leq n-1$) in the QFT(4) accurately. A particular condition $t_{unit} = \tau_{n-1} = t_R$ always achieves all phase rotations in the minimum total time, and then yields the best computing performance for each value of n . The actual time-cost always obeys eq.(8) for any n .

In this way, the actual time-cost varies from $O(n)$ to $O(n2^n)$ for any $n(\leq n_b)$, and follows only the latter for all other values of n . These costs are estimated making the assumption that the QFT is always executed accurately.

An approximate QFT, (AQFT) can reduce the arbitrary numbers of the controlled phase shift gates by sacrificing the accuracy [14]. We can select the AQFT with an actual time-cost between $O(n)$ and $O(n2^n)$, by considering the required accuracy for any n .

We have discussed the actual time-cost from the viewpoint of the phase rotations in the QFT(4). Almost $C_{j,k}(\theta_{k-j})$ operations occur for non-adjacent registers in the QFT. We need to construct such non-adjacent gates

using adjacent ones so here we estimate the actual time-cost required for constructing such non-adjacent gates from adjacent ones. We must construct any non-adjacent gates by use of the adjacent swap technique. The swap $S_{j,k}$ is an operation for exchanging data between two quantum registers simply. It is achieved via the sequence $S_{j,k} = XOR(k,j)XOR(j,k)XOR(k,j)$. The non-adjacent two qubit gate $U_{j,k}$ is achieved by adjacent swaps and adjacent $U_{l,l+1}$ through the following $k-j-1$ sequences.

$$\begin{aligned} U_{j,k} &= S_{k,k-1}S_{k-1,k-2} \cdots S_{j+3,j+2}S_{j+2,j+1}U_{j,j+1} \\ &\quad S_{j+1,j+2}S_{j+2,j+3} \cdots S_{k-2,k-1}S_{k-1,k} \quad (9) \\ &= S_{k,k-1} \cdots S_{l+2,l+1}S_{j,j+1} \cdots S_{l-1,l}U_{l,l+1} \\ &\quad S_{l,l-1} \cdots S_{j+1,j}S_{l+1,l+2} \cdots S_{k-1,k} (j < l < k) \\ &= S_{j,j+1}S_{j+1,j+2}S_{j+2,j+3} \cdots S_{k-2,k-1}U_{k-1,k} \\ &\quad S_{k-1,k-2} \cdots S_{j+3,j+2}S_{j+2,j+1}S_{j+1,j}. \end{aligned} \quad (10)$$

Figure 1 represents the sequence(9). These sequences show that the non-adjacent operation $U_{j,k}$ requires both the time for the adjacent operation $U_{j,j+1}$ and another time in proportion to $|k-j|$ for swaps transferring data.

Considering the construction (9~10), the QFT(4) requires more $(n-1)n(2n-1)/6 \sim O(n^3)$ adjacent swaps for data transfer besides the adjacent controlled phase rotations.

There is, however, a way to reduce some swaps. From sequences(4) and (10), we can obtain the actual quantum circuit shown by Fig.2. In the figure, swaps $S_{j,j+1} \cdots S_{k-1,k}S_{k,k-1} \cdots S_{j+1,j}$ inside each box can be reduced to the identity operation I . The n qubit QFT requires $(n-1)(n-2) \sim O(n^2)$ swaps as a result. It has the same polynomial order as the ideal cost. In this way, there are some cases where the swaps can be reduced due to the symmetry of the algorithm under consideration.

We can conclude that the actual time-cost of the QFT is dominated by the phase rotations and then are $O(n2^n)$ in the range $n > n_b$.

The range of required phase rotation angles increases exponentially with n in the QFT if the accuracy is preserved. In the implementations that are considering, we need to obtain it by controlling the duration or the intensity of the external field. The duration control mode increases the actual time-cost drastically with 2^n . The intensity control mode in BSR takes least time and seems to be the most efficient case for the QFT. This mode, however, leads to another burden on the equipment. The required range of intensity for the applied field increases with 2^n . If the minimum phase rotation is implemented by the field $B = 10^{-3}$ T, the maximum phase rotation for the 100 qubit QFT requires a field intensity $B \sim 10^{27}$ T, which is far beyond the current feasible intensity for the magnetic field.

Our results lead to concerns about the feasibility of factoring huge numbers with polynomial order time-costs in the implementations we are considering.

We have shown that almost all universal gates do not expend the same time in BSR and the SRT. This causes the actual time-cost of the QFT to be drastically different from the ideal one. The ideal cost of the QFT is not effective in the range $n > n_b$ for practical cases of these implementations if the accuracy is preserved.

We believe that both ideal and actual discussions are important for the development of quantum computer science. The former discussion stimulates study of the characteristic of the algorithm itself, and the latter is also important for the current situation in quantum computation as we move from the conceptual system to the physical one. Our discussion shows the necessity of discussing the practical elementary gate in other proposed quantum computers, and of estimating the actual time-cost for other quantum algorithms.

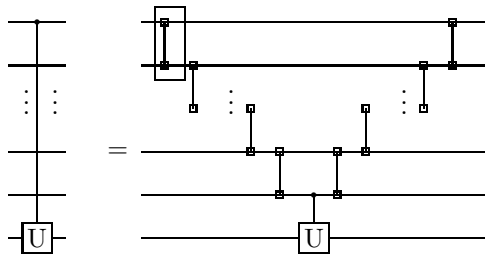


FIG. 1. Non-adjacent operation achieved by utilising adjacent operations. The symbol inside the box represents the adjacent swap $S_{j,k}$.

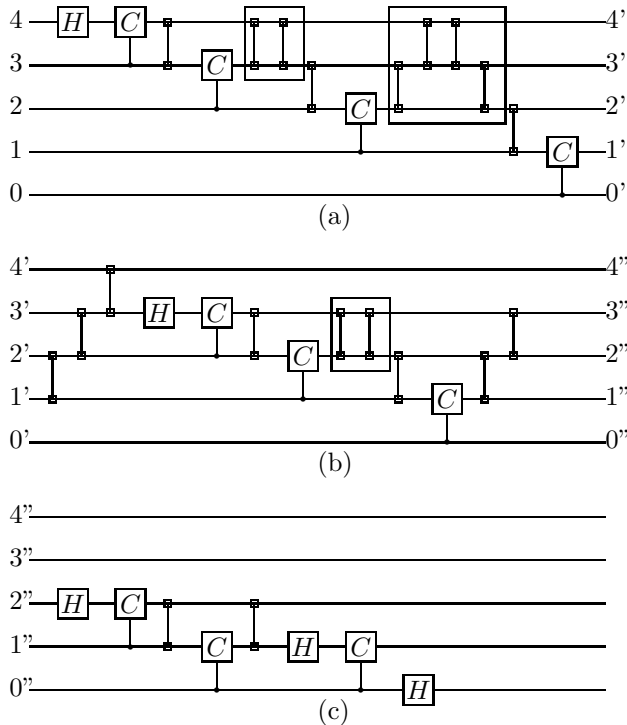


FIG. 2. Figures (a),(b) and (c) in order are the actual quantum circuits of the 5 qubit QFT. This quantum circuit is derived from sequences (4) and (10). The swaps inside the boxes are reducible.

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